67. Assume that $\lim_{x\to a} f(x) = L$. Use Theorem 3.1 to prove that $\lim_{x\to a}[f(x)]^3 = L^3$. Also, show that $\lim_{x\to a}[f(x)]^4 = L^4$.

68. How did you work exercise 67? You probably used Theorem 3.1 to work from $\lim_{x\to a}[f(x)]^2 = L^2$ to $\lim_{x\to a}[f(x)]^3 = L^3$, and then used $\lim_{x\to a}[f(x)]^3 = L^3$ to get $\lim_{x\to a}[f(x)]^4 = L^4$. Going one step at a time, we should be able to reach $\lim_{x\to a}[f(x)]^n = L^n$ for any positive integer n. This is the idea of **mathematical induction.** Formally, we need to show the result is true for a specific value of $n = n_0$ [we show $n_0 = 2$ in the text], then assume the result is true for a general $n = k \geq n_0$. If we show that we can get from the result being true for $n = k$ to the result being true for $n = k + 1$, we have proved that the result is true for any positive integer n. In one sentence, explain why this is true. Use this technique to prove that $\lim_{x\to a}[f(x)]^n = L^n$ for any positive integer n.

69. Find all the errors in the following incorrect string of equalities:

$$\lim_{x\to 0}\frac{1}{x} = \lim_{x\to 0}\frac{x}{x^2} = \lim_{x\to 0} x \lim_{x\to 0}\frac{1}{x^2} = 0 \cdot ? = 0.$$

70. Find all the errors in the following incorrect string of equalities:

$$\lim_{x\to 0}\frac{\sin 2x}{x} = \frac{0}{0} = 1.$$

71. Give an example of functions f and g such that $\lim_{x\to 0}[f(x) + g(x)]$ exists but $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist.

72. Give an example of functions f and g such that $\lim_{x\to 0}[f(x) \cdot g(x)]$ exists but at least one of $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ does not exist.

73. If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ does not exist, is it always true that $\lim_{x\to a}[f(x) + g(x)]$ does not exist? Explain.

74. Is the following true or false? If $\lim_{x\to 0} f(x)$ does not exist, then $\lim_{x\to 0}\frac{1}{f(x)}$ does not exist. Explain.

75. Suppose a state's income tax code states the tax liability on x dollars of taxable income is given by

$$T(x) = \begin{cases} 0.14x & \text{if } 0 \leq x < 10{,}000 \\ 1500 + 0.21x & \text{if } 10{,}000 \leq x \end{cases}.$$

Compute $\lim_{x\to 0^+} T(x)$; why is this good? Compute $\lim_{x\to 10{,}000} T(x)$; why is this bad?

76. Suppose a state's income tax code states that tax liability is 12% on the first \$20,000 of taxable earnings and 16% on the remainder. Find constants a and b for the tax function $T(x) = \begin{cases} a + 0.12x & \text{if } x \leq 20{,}000 \\ b + 0.16(x - 20{,}000) & \text{if } x > 20{,}000 \end{cases}$ such that $\lim_{x\to 0^+} T(x) = 0$ and $\lim_{x\to 20{,}000} T(x)$ exists. Why is it important for these limits to exist?

77. The **greatest integer function** is denoted by $f(x) = [x]$ and equals the greatest integer that is less than or equal to x. Thus, $[2.3] = 2$, $[-1.2] = -2$ and $[3] = 3$. In spite of this last fact, show that $\lim_{x\to 3}[x]$ does not exist.

78. Investigate the existence of (a) $\lim_{x\to 1}[x]$, (b) $\lim_{x\to 1.5}[x]$, (c) $\lim_{x\to 1.5}[2x]$, and (d) $\lim_{x\to 1}(x - [x])$.

EXPLORATORY EXERCISES

1. The value $x = 0$ is called a **zero of multiplicity** $\boldsymbol{n}$ $(n \geq 1)$ for the function f if $\lim_{x\to 0}\frac{f(x)}{x^n}$ exists and is nonzero but $\lim_{x\to 0}\frac{f(x)}{x^{n-1}} = 0$. Show that $x = 0$ is a zero of multiplicity 2 for x^2, $x = 0$ is a zero of multiplicity 3 for x^3 and $x = 0$ is a zero of multiplicity 4 for x^4. For polynomials, what does multiplicity describe? The reason the definition is not as straightforward as we might like is so that it can apply to non-polynomial functions, as well. Find the multiplicity of $x = 0$ for $f(x) = \sin x$; $f(x) = x\sin x$; $f(x) = \sin x^2$. If you know that $x = 0$ is a zero of multiplicity m for $f(x)$ and multiplicity n for $g(x)$, what can you say about the multiplicity of $x = 0$ for $f(x) + g(x)$? $f(x) \cdot g(x)$? $f(g(x))$?

2. We have conjectured that $\lim_{x\to 0}\frac{\sin x}{x} = 1$. Using graphical and numerical evidence, conjecture the value of $\lim_{x\to 0}\frac{\sin 2x}{x}$, $\lim_{x\to 0}\frac{\sin 3x}{x}$, $\lim_{x\to 0}\frac{\sin \pi x}{x}$ and $\lim_{x\to 0}\frac{\sin x/2}{x}$. In general, conjecture the value of $\lim_{x\to 0}\frac{\sin cx}{x}$ for any constant c. Given that $\lim_{x\to 0}\frac{\sin cx}{cx} = 1$ for any constant $c \neq 0$, prove that your conjecture is correct.

1.4 CONTINUITY AND ITS CONSEQUENCES

When you describe something as *continuous,* just what do you have in mind? For example, if told that a machine has been in *continuous* operation for the past 60 hours, most of us would interpret this to mean that the machine has been in operation *all* of that time, without

any interruption at all, even for a moment. Mathematicians mean much the same thing when we say that a function is continuous. A function is said to be *continuous* on an interval if its graph on that interval can be drawn without interruption, that is, without lifting your pencil from the paper.

It is helpful for us to first try to see what it is about the functions whose graphs are shown in Figures 1.22a–1.22d that makes them *discontinuous* (i.e., not continuous) at the point $x = a$.

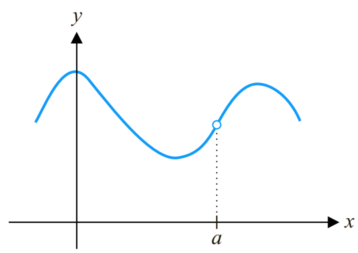


FIGURE 1.22a
$f(a)$ is not defined (the graph has a hole at $x = a$).

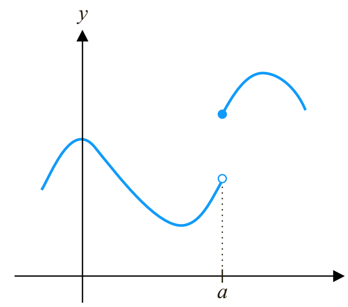


FIGURE 1.22b
$f(a)$ is defined, but $\lim_{x\to a} f(x)$ does not exist (the graph has a jump at $x = a$).

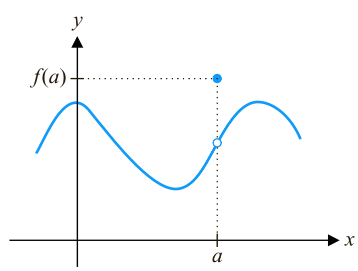


FIGURE 1.22c
$\lim_{x\to a} f(x)$ exists and $f(a)$ is defined, but $\lim_{x\to a} f(x) \neq f(a)$ (the graph has a hole at $x = a$).

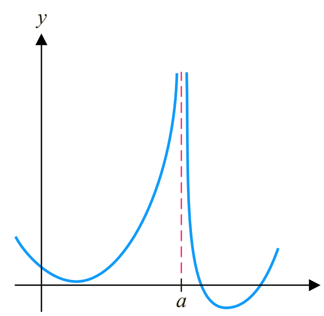


FIGURE 1.22d
$\lim_{x\to a} f(x)$ does not exist (the function "blows up" at $x = a$).

REMARK 4.1

The definition of continuity all boils down to the one condition in (iii), since conditions (i) and (ii) must hold whenever (iii) is met. Further, this says that a function is continuous at a point exactly when you can compute its limit at that point by simply substituting in.

This suggests the following definition of continuity at a point.

DEFINITION 4.1

A function f is **continuous** at $x = a$ when
(i) $f(a)$ is defined, (ii) $\lim_{x\to a} f(x)$ exists and (iii) $\lim_{x\to a} f(x) = f(a)$.
Otherwise, f is said to be **discontinuous** at $x = a$.

For most purposes, it is best for you to think of the intuitive notion of continuity that we've outlined above. Definition 4.1 should then simply follow from your intuitive understanding of the concept.

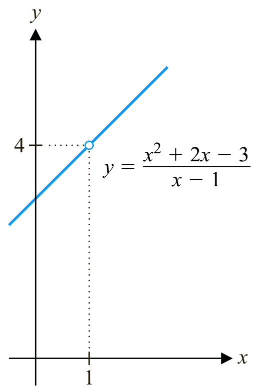


FIGURE 1.23
$y = \dfrac{x^2+2x-3}{x-1}$

REMARK 4.2

You should be careful not to confuse the continuity of a function at a point with its simply being defined there. A function can be defined at a point without being continuous there. (Look back at Figures 1.22b and 1.22c.)

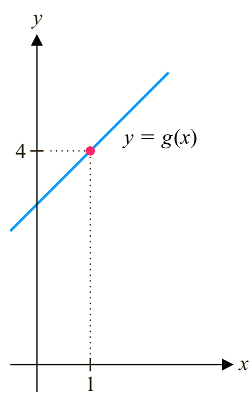


FIGURE 1.24
$y = g(x)$

EXAMPLE 4.1 Finding Where a Rational Function Is Continuous

Determine where $f(x) = \dfrac{x^2+2x-3}{x-1}$ is continuous.

Solution Note that

$$f(x) = \frac{x^2+2x-3}{x-1} = \frac{(x-1)(x+3)}{x-1} \qquad \text{Factoring the numerator.}$$

$$= x+3, \text{ for } x \neq 1. \qquad \text{Canceling common factors.}$$

This says that the graph of f is a straight line, but with a hole in it at $x = 1$, as indicated in Figure 1.23. So, f is discontinuous at $x = 1$, but continuous elsewhere. ■

EXAMPLE 4.2 Removing a Discontinuity

Make the function from example 4.1 continuous everywhere by redefining it at a single point.

Solution In example 4.1, we saw that the function is discontinuous at $x = 1$, since it is undefined there. So, suppose we just go ahead and define it, as follows. Let

$$g(x) = \begin{cases} \dfrac{x^2+2x-3}{x-1}, & \text{if } x \neq 1 \\ a, & \text{if } x = 1, \end{cases}$$

for some real number a.

Notice that $g(x)$ is defined for all x and equals $f(x)$ for all $x \neq 1$. Here, we have

$$\lim_{x\to 1} g(x) = \lim_{x\to 1} \frac{x^2+2x-3}{x-1}$$

$$= \lim_{x\to 1} (x+3) = 4.$$

Observe that if we choose $a = 4$, we now have that

$$\lim_{x\to 1} g(x) = 4 = g(1)$$

and so, g is continuous at $x = 1$.

Note that the graph of g is the same as the graph of f seen in Figure 1.23, except that we now include the point (1, 4) (see Figure 1.24). Also, note that there's a very simple way to write $g(x)$. (Think about this.) ■

When we can remove a discontinuity by redefining the function at that point, we call the discontinuity **removable.** Not all discontinuities are removable, however. Carefully examine Figures 1.22a–1.22d and convince yourself that the discontinuities in Figures 1.22a and 1.22c are removable, while those in Figures 1.22b and 1.22d are nonremovable. Briefly, a function f has a removable discontinuity at $x = a$ if $\lim_{x\to a} f(x)$ exists and either $f(a)$ is undefined or $\lim_{x\to a} f(x) \neq f(a)$.

EXAMPLE 4.3 Nonremovable Discontinuities

Find all discontinuities of $f(x) = \dfrac{1}{x^2}$ and $g(x) = \cos\left(\dfrac{1}{x}\right)$.

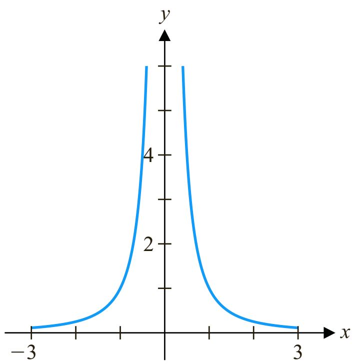


FIGURE 1.25a
$y = \frac{1}{x^2}$

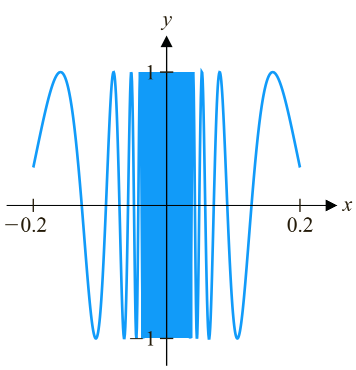


FIGURE 1.25b
$y = \cos(1/x)$

Solution You should observe from Figure 1.25a (also, construct a table of function values) that

$$\lim_{x\to 0} \frac{1}{x^2} \text{ does not exist.}$$

Hence, f is discontinuous at $x = 0$.

Similarly, observe that $\lim_{x\to 0} \cos(1/x)$ does not exist, due to the endless oscillation of $\cos(1/x)$ as x approaches 0 (see Figure 1.25b).

In both cases, notice that since the limits do not exist, there is no way to redefine either function at $x = 0$ to make it continuous there. ■

From your experience with the graphs of some common functions, the following result should come as no surprise.

THEOREM 4.1

All polynomials are continuous everywhere. Additionally, $\sin x$, $\cos x$, $\tan^{-1} x$ and e^x are continuous everywhere, $\sqrt[n]{x}$ is continuous for all x, when n is odd and for $x > 0$, when n is even. We also have $\ln x$ is continuous for $x > 0$ and $\sin^{-1} x$ and $\cos^{-1} x$ are continuous for $-1 < x < 1$.

PROOF

We have already established (in Theorem 3.2) that for any polynomial $p(x)$ and any real number a,

$$\lim_{x\to a} p(x) = p(a),$$

from which it follows that p is continuous at $x = a$. The rest of the theorem follows from Theorem 3.4 in a similar way. ■

From these very basic continuous functions, we can build a large collection of continuous functions, using Theorem 4.2.

THEOREM 4.2

Suppose that f and g are continuous at $x = a$. Then all of the following are true:

(i) $(f \pm g)$ is continuous at $x = a$,
(ii) $(f \cdot g)$ is continuous at $x = a$ and
(iii) (f/g) is continuous at $x = a$ if $g(a) \neq 0$.

Simply put, Theorem 4.2 says that a sum, difference or product of continuous functions is continuous, while the quotient of two continuous functions is continuous at any point at which the denominator is nonzero.

PROOF

(i) If f and g are continuous at $x = a$, then

$$\begin{aligned}\lim_{x\to a}[f(x) \pm g(x)] &= \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) && \text{From Theorem 3.1.}\\ &= f(a) \pm g(a) && \text{Since } f \text{ and } g \text{ are continuous at } a.\\ &= (f \pm g)(a),\end{aligned}$$

by the usual rules of limits. Thus, $(f \pm g)$ is also continuous at $x = a$.

Parts (ii) and (iii) are proved in a similar way and are left as exercises. ■

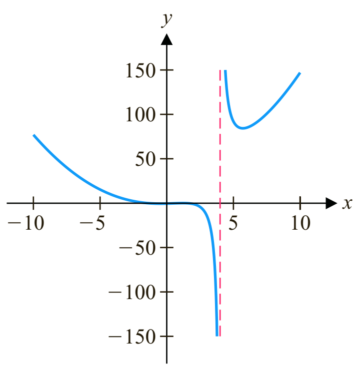


FIGURE 1.26

$$y = \frac{x^4 - 3x^2 + 2}{x^2 - 3x - 4}$$

EXAMPLE 4.4 Continuity for a Rational Function

Determine where f is continuous, for $f(x) = \dfrac{x^4 - 3x^2 + 2}{x^2 - 3x - 4}$.

Solution Here, f is a quotient of two polynomial (hence continuous) functions. The graph of the function indicated in Figure 1.26 suggests a vertical asymptote at around $x = 4$, but doesn't indicate any other discontinuity. From Theorem 4.2, f will be continuous at all x where the denominator is not zero, that is, where

$$x^2 - 3x - 4 = (x + 1)(x - 4) \neq 0.$$

Thus, f is continuous for $x \neq -1, 4$. (Think about why you didn't see anything peculiar about the graph at $x = -1$.) ■

With the addition of the result in Theorem 4.3, we will have all the basic tools needed to establish the continuity of most elementary functions.

THEOREM 4.3

Suppose that $\lim_{x\to a} g(x) = L$ and f is continuous at L. Then,

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(L).$$

A proof of Theorem 4.3 is given in Appendix A.

Notice that this says that if f is continuous, then we can bring the limit "inside." This should make sense, since as $x \to a$, $g(x) \to L$ and so, $f(g(x)) \to f(L)$, since f is continuous at L.

COROLLARY 4.1

Suppose that g is continuous at a and f is continuous at $g(a)$. Then, the composition $f \circ g$ is continuous at a.

PROOF

From Theorem 4.3, we have

$$\begin{aligned}\lim_{x\to a}(f\circ g)(x) &= \lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)\\ &= f(g(a)) = (f\circ g)(a). \quad \text{Since } g \text{ is continuous at } a. \blacksquare\end{aligned}$$

EXAMPLE 4.5 Continuity for a Composite Function

Determine where $h(x) = \cos(x^2 - 5x + 2)$ is continuous.

Solution Note that

$$h(x) = f(g(x)),$$

where $g(x) = x^2 - 5x + 2$ and $f(x) = \cos x$. Since both f and g are continuous for all x, h is continuous for all x, by Corollary 4.1. ■

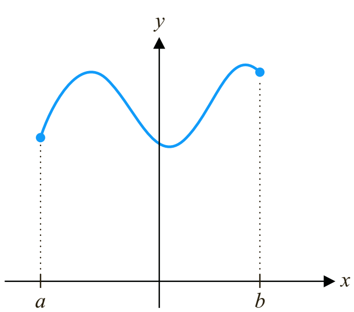


FIGURE 1.27
f continuous on $[a, b]$

DEFINITION 4.2

If f is continuous at every point on an open interval (a, b), we say that f is **continuous on (a, b).** Following Figure 1.27, we say that f is **continuous on the closed interval $[a, b]$,** if f is continuous on the open interval (a, b) and

$$\lim_{x\to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x\to b^-} f(x) = f(b).$$

Finally, if f is continuous on all of $(-\infty, \infty)$, we simply say that f is **continuous.** (That is, when we don't specify an interval, we mean continuous everywhere.)

For many functions, it's a simple matter to determine the intervals on which the function is continuous. We illustrate this in example 4.6.

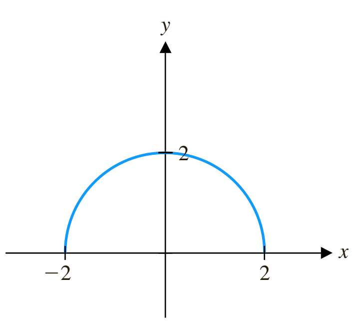


FIGURE 1.28
$y = \sqrt{4 - x^2}$

EXAMPLE 4.6 Continuity on a Closed Interval

Determine the interval(s) where f is continuous, for $f(x) = \sqrt{4 - x^2}$.

Solution First, observe that f is defined only for $-2 \le x \le 2$. Next, note that f is the composition of two continuous functions and hence, is continuous for all x for which $4 - x^2 > 0$. We show a graph of the function in Figure 1.28. Since

$$4 - x^2 > 0$$

for $-2 < x < 2$, we have that f is continuous for all x in the interval $(-2, 2)$, by Theorem 4.1 and Corollary 4.1. Finally, we test the endpoints to see that $\lim_{x\to 2^-} \sqrt{4 - x^2} = 0 = f(2)$ and $\lim_{x\to -2^+} \sqrt{4 - x^2} = 0 = f(-2)$, so that f is continuous on the closed interval $[-2, 2]$. ■

EXAMPLE 4.7 Interval of Continuity for a Logarithm

Determine the interval(s) where $f(x) = \ln(x - 3)$ is continuous.

Solution It follows from Theorem 4.1 and Corollary 4.1 that f is continuous whenever $(x - 3) > 0$ (i.e., for $x > 3$). Thus, f is continuous on the interval $(3, \infty)$. ■

The Internal Revenue Service presides over some of the most despised functions in existence. Look up the current Tax Rate Schedules. In 2002, the first few lines (for single taxpayers) looked like:

For taxable amount over	*but not over*	*your tax liability is*	*minus*
\$0	\$6000	10%	\$0
\$6000	\$27,950	15%	\$300
\$27,950	\$ 67,700	27%	\$3654

Where do the numbers \$300 and \$3654 come from? If we write the tax liability $T(x)$ as a function of the taxable amount x (assuming that x can be any real value and not just a whole dollar amount), we have

$$T(x) = \begin{cases} 0.10x & \text{if } 0 < x \le 6000 \\ 0.15x - 300 & \text{if } 6000 < x \le 27{,}950 \\ 0.27x - 3654 & \text{if } 27{,}950 < x \le 67{,}700. \end{cases}$$

Be sure you understand our translation so far. Note that it is important that this be a continuous function: think of the fairness issues that would arise if it were not!

EXAMPLE 4.8 Continuity of Federal Tax Tables

Verify that the federal tax rate function $T(x)$ is continuous at the "joint" $x = 27{,}950$. Then, find a to complete the table. (You will find b and c as exercises.)

For taxable amount over	*but not over*	*your tax liability is*	*minus*
\$67,700	\$141,250	30%	a
\$141,250	\$307,050	35%	b
\$307,050	—	38.6%	c

Solution For $T(x)$ to be continuous at $x = 27{,}950$, we must have

$$\lim_{x \to 27{,}950^-} T(x) = \lim_{x \to 27{,}950^+} T(x).$$

Since both functions $0.15x - 300$ and $0.27x - 3654$ are continuous, we can compute the one-sided limits by substituting $x = 27{,}950$. Thus,

$$\lim_{x \to 27{,}950^-} T(x) = 0.15(27{,}950) - 300 = 3892.50$$

and

$$\lim_{x \to 27{,}950^+} T(x) = 0.27(27{,}950) - 3654 = 3892.50.$$

Since the one-sided limits agree and equal the value of the function at that point, $T(x)$ is continuous at $x = 27{,}950$. We leave it as an exercise to establish that $T(x)$ is also continuous at $x = 6000$. (It's worth noting that the function could be written with equal signs on all of the inequalities; this would be incorrect if the function were discontinuous.) To complete the table, we choose a to get the one-sided limits at $x = 67{,}700$ to match. We have

$$\lim_{x \to 67{,}700^-} T(x) = 0.27(67{,}700) - 3654 = 14{,}625,$$

while $$\lim_{x \to 67{,}700^+} T(x) = 0.30(67{,}700) - a = 20{,}310 - a.$$

So, we set the one-sided limits equal, to obtain

$$14{,}625 = 20{,}310 - a$$

or $$a = 20{,}310 - 14{,}625 = 5685.$$

HISTORICAL NOTES

Karl Weierstrass (1815–1897) A German mathematician who proved the Intermediate Value Theorem and several other fundamental results of the calculus. Weierstrass was known as an excellent teacher whose students circulated his lecture notes throughout Europe, because of their clarity and originality. Also known as a superb fencer, Weierstrass was one of the founders of modern mathematical analysis.

Theorem 4.4 should seem an obvious consequence of our intuitive definition of continuity.

THEOREM 4.4 (Intermediate Value Theorem)

Suppose that f is continuous on the closed interval $[a, b]$ and W is any number between $f(a)$ and $f(b)$. Then, there is a number $c \in [a, b]$ for which $f(c) = W$.

Theorem 4.4 says that if f is continuous on $[a, b]$, then f must take on *every* value between $f(a)$ and $f(b)$ at least once. That is, a continuous function cannot skip over any numbers between its values at the two endpoints. To do so, the graph would need to leap across the horizontal line $y = W$, something that continuous functions cannot do (see Figure 1.29a). Of course, a function may take on a given value W more than once (see Figure 1.29b). We must point out that, although these graphs make this result seem reasonable, like any other result, Theorem 4.4 requires proof. The proof is more complicated than you might imagine and we must refer you to an advanced calculus text.

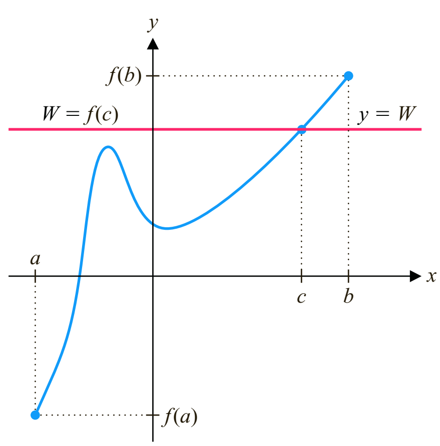


FIGURE 1.29a
An illustration of the Intermediate Value Theorem

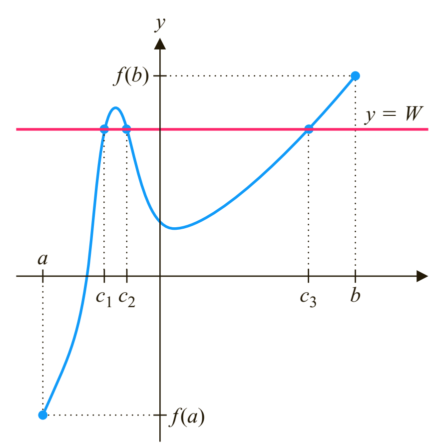


FIGURE 1.29b
More than one value of c

In Corollary 4.2, we see an immediate and useful application of the Intermediate Value Theorem.

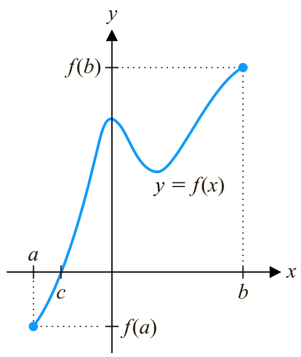


FIGURE 1.30
Intermediate Value Theorem where c is a zero of f

COROLLARY 4.2

Suppose that f is continuous on $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs [i.e., $f(a) \cdot f(b) < 0$]. Then, there is at least one number $c \in (a, b)$ for which $f(c) = 0$. (Recall that c is then a *zero* of f.)

Notice that Corollary 4.2 is simply the special case of the Intermediate Value Theorem where $W = 0$ (see Figure 1.30). The Intermediate Value Theorem and Corollary 4.2 are examples of *existence theorems;* they tell you that there *exists* a number c satisfying some condition, but they do *not* tell you what c is.

The Method of Bisections

In example 4.9, we see how Corollary 4.2 can help us locate the zeros of a function.

EXAMPLE 4.9 Finding Zeros by the Method of Bisections

Find the zeros of $f(x) = x^5 + 4x^2 - 9x + 3$.

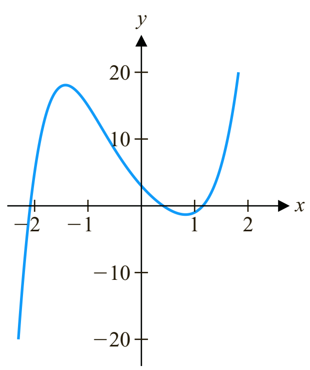


FIGURE 1.31
$y = x^5 + 4x^2 - 9x + 3$

Solution If f were a quadratic polynomial, you could certainly find its zeros. However, you don't have any formulas for finding zeros of polynomials of degree 5. The only alternative is to approximate the zeros. A good starting place would be to draw a graph of $y = f(x)$ like the one in Figure 1.31. There are three zeros visible on the graph. Since f is a polynomial, it is continuous everywhere and so, Corollary 4.2 says that there must be a zero on any interval on which the function changes sign. From the graph, you can see that there must be zeros between -3 and -2, between 0 and 1 and between 1 and 2. We could also conclude this by computing say, $f(0) = 3$ and $f(1) = -1$. Although we've now found intervals that contain zeros, the question remains as to how we can *find* the zeros themselves.

While a rootfinding program can provide an accurate approximation, the issue here is not so much to get an answer as it is to understand how to find one. We suggest a simple yet effective method, called the **method of bisections.**

For the zero between 0 and 1, a reasonable guess might be the midpoint, 0.5. Since $f(0.5) \approx -0.469 < 0$ and $f(0) = 3 > 0$, there must be a zero between 0 and 0.5. Next, the midpoint of $[0, 0.5]$ is 0.25 and $f(0.25) \approx 1.001 > 0$, so that the zero is on the interval $(0.25, 0.5)$. We continue in this way to narrow the interval on which there's a zero until the interval is sufficiently small so that any point in the interval can serve as an adequate approximation to the actual zero. We do this in the following table.

a	*b*	*f(a)*	*f(b)*	*Midpoint*	*f(midpoint)*
0	1	3	−1	0.5	−0.469
0	0.5	3	−0.469	0.25	1.001
0.25	0.5	1.001	−0.469	0.375	0.195
0.375	0.5	0.195	−0.469	0.4375	−0.156
0.375	0.4375	0.195	−0.156	0.40625	0.015
0.40625	0.4375	0.015	−0.156	0.421875	−0.072
0.40625	0.421875	0.015	−0.072	0.4140625	−0.029
0.40625	0.4140625	0.015	−0.029	0.41015625	−0.007
0.40625	0.41015625	0.015	−0.007	0.408203125	0.004

If you continue this process through 20 more steps, you ultimately arrive at the approximate zero $x = 0.40892288$, which is accurate to at least eight decimal places. ■

This method of bisections is a tedious process, if you're working it with pencil and paper. It is interesting because it's reliable and it's a simple, yet general method for finding approximate zeros. Computer and calculator rootfinding utilities are very useful, but our purpose here is to provide you with an understanding of how basic rootfinding works. We discuss a more powerful method for finding roots in Chapter 3.

EXERCISES 1.4

WRITING EXERCISES

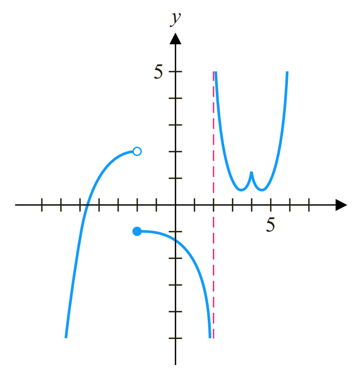
1. Think about the following "real-life" functions, each of which is a function of the independent variable time: the height of a falling object, the velocity of an object, the amount of money in a bank account, the cholesterol level of a person, the heart rate of a person, the amount of a certain chemical present in a test tube and a machine's most recent measurement of the cholesterol level of a person. Which of these are continuous functions? For each function you identify as discontinuous, what is the real-life meaning of the discontinuities?

2. Whether a process is continuous or not is not always clear-cut. When you watch television or a movie, the action seems to be continuous. This is an optical illusion, since both movies and television consist of individual "snapshots" that are played back at many frames per second. Where does the illusion of continuous motion come from? Given that the average person blinks several times per minute, is our perception of the world actually continuous? (In what cognitive psychologists call **temporal binding,** the human brain first decides whether a stimulus is important enough to merit conscious consideration. If so, the brain "predates" the stimulus so that the person correctly identifies when the stimulus actually occurred.)

3. When you sketch the graph of the parabola $y = x^2$ with pencil or pen, is your sketch (at the molecular level) actually the graph of a continuous function? Is your calculator or computer's graph actually the graph of a continuous function? On many calculators, you have the option of a connected or disconnected graph. At the pixel level, does a connected graph show the graph of a function? Does a disconnected graph show the graph of a continuous function? Do we ever have problems correctly interpreting a graph due to these limitations? In the exercises in section 1.7, we examine one case where our perception of a computer graph depends on which choice is made.

4. For each of the graphs in Figures 1.22a–1.22d, describe (with an example) what the formula for $f(x)$ might look like to produce the given discontinuity.

In exercises 1–6, use the given graph to identify all discontinuities of the functions.

1.


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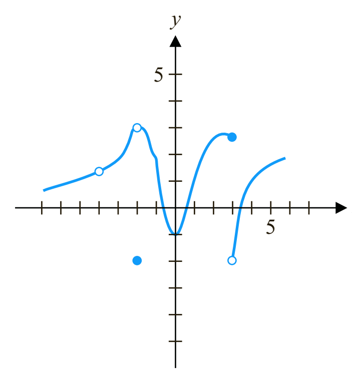

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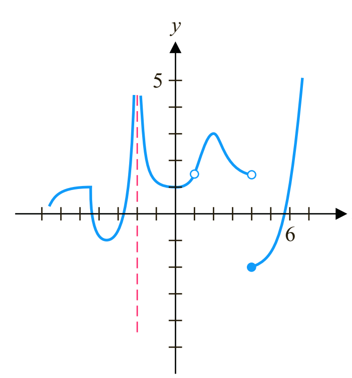

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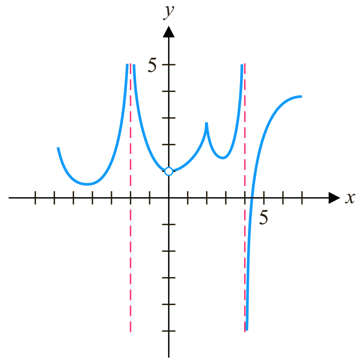

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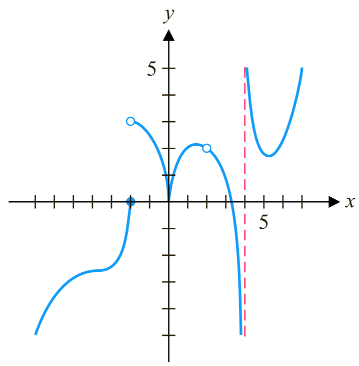

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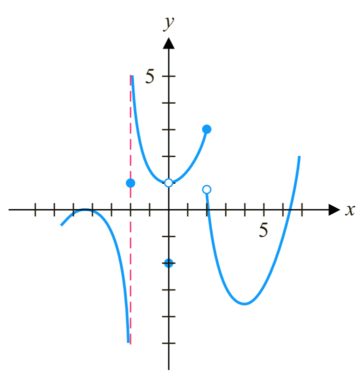


In exercises 7–12, explain why each function is discontinuous at the given point by indicating which of the three conditions in Definition 4.1 are not met.

7. $f(x) = \dfrac{x}{x-1}$ at $x = 1$

8. $f(x) = \dfrac{x^2-1}{x-1}$ at $x = 1$

9. $f(x) = \sin \dfrac{1}{x}$ at $x = 0$

10. $f(x) = e^{1/x}$ at $x = 0$

11. $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3 & \text{if } x = 2 \\ 3x-2 & \text{if } x > 2 \end{cases}$ at $x = 2$

12. $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3x-2 & \text{if } x > 2 \end{cases}$ at $x = 2$

In exercises 13–24, find all discontinuities of $f(x)$. For each discontinuity that is removable, define a new function that removes the discontinuity.

13. $f(x) = \dfrac{x-1}{x^2-1}$

14. $f(x) = \dfrac{4x}{x^2+x-2}$

15. $f(x) = \dfrac{4x}{x^2+4}$

16. $f(x) = \dfrac{3x}{x^2-2x-4}$

17. $f(x) = x^2 \tan x$

18. $f(x) = x \cot x$

19. $f(x) = x \ln x^2$

20. $f(x) = e^{-4/x^2}$

21. $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ x^2 & \text{if } x \ge 1 \end{cases}$

22. $f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \ne 0 \\ 1 & \text{if } x = 0 \end{cases}$

23. $f(x) = \begin{cases} 3x-1 & \text{if } x \le -1 \\ x^2+5x & \text{if } -1 < x < 1 \\ 3x^3 & \text{if } x \ge 1 \end{cases}$

24. $f(x) = \begin{cases} 2x & \text{if } x < 0 \\ \sin x & \text{if } 0 < x \le \pi \\ x-\pi & \text{if } x > \pi \end{cases}$

In exercises 25–32, determine the intervals on which $f(x)$ is continuous.

25. $f(x) = \sqrt{x+3}$

26. $f(x) = \sqrt{x^2-4}$

27. $f(x) = \sqrt[3]{x+2}$

28. $f(x) = (x-1)^{3/2}$

29. $f(x) = \sin(x^2+2)$

30. $f(x) = \cos\left(\dfrac{1}{x}\right)$

31. $f(x) = \ln(x+1)$

32. $f(x) = \ln(4-x^2)$

In exercises 33–35, determine values of a and b that make the given function continuous.

33. $f(x) = \begin{cases} \dfrac{2\sin x}{x} & \text{if } x < 0 \\ a & \text{if } x = 0 \\ b\cos x & \text{if } x > 0 \end{cases}$

34. $f(x) = \begin{cases} ae^x + 1 & \text{if } x < 0 \\ \sin^{-1}\dfrac{x}{2} & \text{if } 0 \le x \le 2 \\ x^2 - x + b & \text{if } x > 2 \end{cases}$

35. $f(x) = \begin{cases} a(\tan^{-1} x + 2) & \text{if } x < 0 \\ 2e^{bx} + 1 & \text{if } 0 \le x \le 3 \\ \ln(x-2) + x^2 & \text{if } x > 3 \end{cases}$

36. Prove Corollary 4.1.

37. Suppose that a state's income tax code states that the tax liability on x dollars of taxable income is given by

$$T(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 0.14x & \text{if } 0 < x < 10{,}000 \\ c + 0.21x & \text{if } 10{,}000 \le x. \end{cases}$$

Determine the constant c that makes this function continuous for all x. Give a rationale why such a function should be continuous.

38. Suppose a state's income tax code states that tax liability is 12% on the first \$20,000 of taxable earnings and 16% on the remainder. Find constants a and b for the tax function

$$T(x) = \begin{cases} 0 & \text{if } x \le 0 \\ a + 0.12x & \text{if } 0 < x \le 20{,}000 \\ b + 0.16(x - 20{,}000) & \text{if } x > 20{,}000 \end{cases}$$

such that $T(x)$ is continuous for all x.

39. In example 4.8, find b and c to complete the table.

40. In example 4.8, show that $T(x)$ is continuous for $x = 6000$.

In exercises 41–46, use the Intermediate Value Theorem to verify that $f(x)$ has a zero in the given interval. Then use the method of bisections to find an interval of length 1/32 that contains the zero.

41. $f(x) = x^2 - 7$, $[2, 3]$

42. $f(x) = x^3 - 4x - 2$, $[2, 3]$

43. $f(x) = x^3 - 4x - 2$, $[-1, 0]$

44. $f(x) = x^3 - 4x - 2$, $[-2, -1]$

45. $f(x) = \cos x - x$, $[0, 1]$

46. $f(x) = e^x + x$, $[-1, 0]$

A function is continuous from the right at $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$. In exercises 47–50, determine whether $f(x)$ is continuous from the right at $x = 2$.

47. $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3x - 1 & \text{if } x \ge 2 \end{cases}$

48. $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3 & \text{if } x = 2 \\ 3x - 3 & \text{if } x > 2 \end{cases}$

49. $f(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ 3x - 3 & \text{if } x > 2 \end{cases}$

50. $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3x - 2 & \text{if } x > 2 \end{cases}$

51. Define what it means for a function to be **continuous from the left** at $x = a$ and determine which of the functions in exercises 47–50 are continuous from the left at $x = 2$.

52. Suppose that $f(x) = \dfrac{g(x)}{h(x)}$ and $h(a) = 0$. Determine whether each of the following statements is always true, always false, or maybe true/maybe false. Explain. (a) $\lim_{x \to a} f(x)$ does not exist. (b) $f(x)$ is discontinuous at $x = a$.

53. The sex of newborn Mississippi alligators is determined by the temperature of the eggs in the nest. The eggs fail to develop unless the temperature is between 26°C and 36°C. All eggs between 26°C and 30°C develop into females, and eggs between 34°C and 36°C develop into males. The percentage of females decreases from 100% at 30°C to 0% at 34°C. If $f(T)$ is the percentage of females developing from an egg at T°C, then

$$f(T) = \begin{cases} 100 & \text{if } 26 \le T \le 30 \\ g(T) & \text{if } 30 < T < 34 \\ 0 & \text{if } 34 \le T \le 36, \end{cases}$$

for some function $g(T)$. Explain why it is reasonable that $f(T)$ be continuous. Determine a function $g(T)$ such that $0 \le g(T) \le 100$ for $30 \le T \le 34$ and the resulting function $f(T)$ is continuous. [Hint: It may help to draw a graph first and make $g(T)$ linear.]

54. If $f(x) = \begin{cases} x^2, & \text{if } x \ne 0 \\ 4, & \text{if } x = 0 \end{cases}$ and $g(x) = 2x$, show that

$$\lim_{x \to 0} f(g(x)) \ne f\left(\lim_{x \to 0} g(x)\right).$$

55. If you push on a large box resting on the ground, at first nothing will happen because of the static friction force that opposes motion. If you push hard enough, the box will start sliding, although there is again a friction force that opposes the motion. Suppose you are given the following description of the friction force. Up to 100 pounds, friction matches the force you apply to the box. Over 100 pounds, the box will move and the friction force will equal 80 pounds. Sketch a graph of friction as a function of your applied force based on this description. Where is this graph discontinuous? What is significant physically about this point? Do you think the friction force actually ought to be continuous? Modify the graph to make it continuous while still retaining most of the characteristics described.

56. For $f(x) = 2x - \dfrac{400}{x}$, we have $f(-1) > 0$ and $f(2) < 0$. Does the Intermediate Value Theorem guarantee a zero of $f(x)$ between $x = -1$ and $x = 2$? What happens if you try the method of bisections?

57. On Monday morning, a saleswoman leaves on a business trip at 7:13 A.M. and arrives at her destination at 2:03 P.M. The following morning, she leaves for home at 7:17 A.M. and arrives at 1:59 P.M. The woman notices that at a particular stoplight along the way, a nearby bank clock changes from 10:32 A.M. to 10:33 A.M. on both days. Therefore, she must have been at the same location at the same time on both days. Her boss doesn't believe that such an unlikely coincidence could occur. Use the Intermediate Value Theorem to argue that it *must* be true that at some point on the trip, the saleswoman was at exactly the same place at the same time on both Monday and Tuesday.

58. Suppose you ease your car up to a stop sign at the top of a hill. Your car rolls back a couple of feet and then you drive through

the intersection. A police officer pulls you over for not coming to a complete stop. Use the Intermediate Value Theorem to argue that there was an instant in time when your car was stopped (in fact, there were at least two). What is the difference between this stopping and the stopping that the police officer wanted to see?

59. Suppose a worker's salary starts at \$40,000 with \$2000 raises every 3 months. Graph the salary function $s(t)$; why is it discontinuous? How does the function $f(t) = 40{,}000 + \frac{2000}{3}t$ (t in months) compare? Why might it be easier to do calculations with $f(t)$ than $s(t)$?

60. Prove the final two parts of Theorem 4.2.

61. Suppose that $f(x)$ is a continuous function with consecutive zeros at $x = a$ and $x = b$; that is, $f(a) = f(b) = 0$ and $f(x) \neq 0$ for $a < x < b$. Further, suppose that $f(c) > 0$ for some number c between a and b. Use the Intermediate Value Theorem to argue that $f(x) > 0$ for all $a < x < b$.

62. Use the method of bisections to estimate the other two zeros in example 4.9.

63. Suppose that $f(x)$ is continuous at $x = 0$. Prove that $\lim_{x\to 0} x f(x) = 0$.

64. The **converse** of exercise 63 is not true. That is, the fact $\lim_{x\to 0} x f(x) = 0$ does not guarantee that $f(x)$ is continuous at $x = 0$. Find a counterexample; that is, find a function f such that $\lim_{x\to 0} x f(x) = 0$ and $f(x)$ is not continuous at $x = 0$.

65. If $f(x)$ is continuous at $x = a$, prove that $g(x) = |f(x)|$ is continuous at $x = a$.

66. Determine whether the converse of exercise 65 is true. That is, if $|f(x)|$ is continuous at $x = a$, is it necessarily true that $f(x)$ must be continuous at $x = a$?

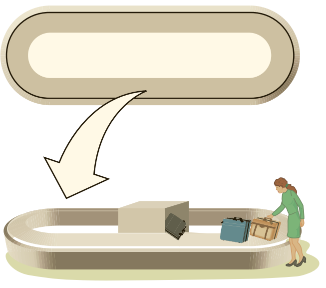
67. Let $f(x)$ be a continuous function for $x \geq a$ and define $h(x) = \max_{a \leq t \leq x} f(t)$. Prove that $h(x)$ is continuous for $x \geq a$. Would this still be true without the assumption that $f(x)$ is continuous?

68. Graph $f(x) = \dfrac{\sin|x^3 - 3x^2 + 2x|}{x^3 - 3x^2 + 2x}$ and determine all discontinuities.

EXPLORATORY EXERCISES

1. In the text, we discussed the use of the method of bisections to find an approximate solution of equations such as $f(x) = x^3 + 5x - 1 = 0$. We can start by noticing that $f(0) = -1$ and $f(1) = 5$. Since $f(x)$ is continuous, the Intermediate Value Theorem tells us that there is a solution between $x = 0$ and $x = 1$. For the method of bisections, we guess the midpoint, $x = 0.5$. Is there any reason to suspect that the solution is actually closer to $x = 0$ than to $x = 1$? Using the function values $f(0) = -1$ and $f(1) = 5$, devise your own method of guessing the location of the solution. Generalize your method to using $f(a)$ and $f(b)$, where one function value is positive and one is negative. Compare your method to the method of bisections on the problem $x^3 + 5x - 1 = 0$; for both methods, stop when you are within 0.001 of the solution, $x \approx 0.198437$. Which method performed better? Before you get overconfident in your method, compare the two methods again on $x^3 + 5x^2 - 1 = 0$. Does your method get close on the first try? See if you can determine graphically why your method works better on the first problem.

2. You have probably seen the turntables on which luggage rotates at the airport. Suppose that such a turntable has two long straight parts with a semicircle on each end (see the figure). We will model the left/right movement of the luggage. Suppose the straight part is 40 ft long, extending from $x = -20$ to $x = 20$. Assume that our luggage starts at time $t = 0$ at location $x = -20$, and that it takes 60 s for the luggage to reach $x = 20$. Suppose the radius of the circular portion is 5 ft and it takes the luggage 30 s to complete the half-circle. We model the straight-line motion with a linear function $x(t) = at + b$. Find constants a and b so that $x(0) = -20$ and $x(60) = 20$. For the circular motion, we use a cosine (Why is this a good choice?) $x(t) = 20 + d \cdot \cos(et + f)$ for constants d, e and f. The requirements are $x(60) = 20$ (since the motion is continuous), $x(75) = 25$ and $x(90) = 20$. Find values of d, e and f to make this work. Find equations for the position of the luggage along the backstretch and the other semicircle. What would the motion be from then on?



Luggage carousel

3. Determine all x's for which each function is continuous.

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ x & \text{if } x \text{ is rational} \end{cases},$$

$$g(x) = \begin{cases} x^2 + 3 & \text{if } x \text{ is irrational} \\ 4x & \text{if } x \text{ is rational} \end{cases} \text{ and}$$

$$h(x) = \begin{cases} \cos 4x & \text{if } x \text{ is irrational} \\ \sin 4x & \text{if } x \text{ is rational} \end{cases}.$$